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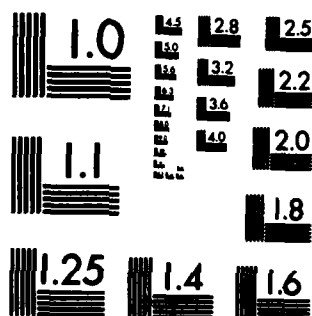FILMED

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ON COLLOCATION SCHEMES FOR
QUASILINEAR SINGULARLY PERTURBED
BOUNDARY VALUE PROBLEMS

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ON COLLOCATION SCHEMES FOR QUASILINEAR
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ABSTRACT

A numerical method for boundary value problems for quasilinear systems of

singularly perturbed ordinary differential equations is presented. The method

is based on collocation with polynomial splines. The stability properties of

the associated difference operator are examined and a stepsize algorithm to

achieve a certain over-all accuracy is developed. The number of gridpoints

required by the algorithm is estimated.

AMS (MOS) Subject Classifications: 34E99, 65L10

Key Words: Collocation Methods, Singular Perturbations, Boundary Value

            Problems

Work Unit Number 3 - Numerical Analysis

SIGNIFICANCE AND EXPLANATION

Many high-order discretization methods for the solution of two-point
boundary value problems for systems of ordinary differential equations are
already circulated. However, these methods can behave quite poorly in case
the solution has large derivatives, unless severe restrictions on the mesh are
imposed. The reason for these restrictions is mainly a stability problem.

In this paper a strongly A-stable difference method based on polynomial
collocation is developed for a class of quasilinear, singularly perturbed,
two-point boundary value problems. Many problems of practical interest are
included in this class, for instance the nonlinear deformation of thin beams
or one-dimensional models of carrier transport in semiconductor devices. The
method combines the advantages of having the same stability properties as the
lower order methods which are used already, with the high order of convergence
of collocation methods. It is shown that the number of gridpoints (and
therefore the amount of computing time and required storage) is of the same
order of magnitude as the one required for solving unperturbed problems.

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ON COLLOCATION SCHEMES FOR QUASILINEAR
SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

CHRISTIAN RINGHOFER

1. Introduction.

We consider quasilinear systems of singularly perturbed O.D.E.'s on the interval

$[0,1]$ of the form

$$F_\varepsilon(y,z) \; = \; 0 \; , \quad b_\varepsilon(y,z) \; = \; 0$$

(1.1)
$$F_\varepsilon(y,z) \; := \; \begin{pmatrix} \varepsilon y'(t,\varepsilon) - f(t,y,z,\varepsilon) \\ z'(t,\varepsilon) - g(t,y,z,\varepsilon) \end{pmatrix}$$

$$b_\varepsilon(y,z) \; = \; b(y(0,\varepsilon),y(1,\varepsilon),\; z(0,\varepsilon),\; z(1,\varepsilon),\varepsilon)$$

y and z are vectors of dimension n and m. The prime denotes the derivative with

respect to t.   (1.1) is quasilinear in the following sense:

$$f(t,y,z,\varepsilon) \; = \; f_1(t,z)y + f_2(t,y,z,\varepsilon)$$

(1.2)

$$b(y_0,y_1,z_0,z_1,\varepsilon) \; = \; b_1(z_0,z_1)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + b_2(y_0,y_1,z_0,z_1,\varepsilon)$$

f_1 and b_1 are matrices of appropriate dimensions. We assume that the derivatives $\dfrac{\partial f_2}{\partial y}$

and $\dfrac{\partial b_2}{\partial y_0, y_1}$ are uniformly of order ε for $y,z,y_0,y_1\; z_0,z_1$ in any bounded domain of the

appropriate real spaces. For f_2,b_2 and g there exist asymptotic expansions in powers

of ε. The $n \times n$-matrix f_1 is a block-diagonal-matrix of the form

(1.3)
$$f_1(t,z) \; = \; \begin{pmatrix} f_1^-(t,z) & 0 \\ 0 & f_1^+(t,z) \end{pmatrix} \quad .$$

The square-matrices f_1^- and f_1^+ are of dimension n_- and n_+ (where $n_- + n_+ = n$

holds). f_1^- has only strictly stable and f_2^+ has only strictly unstable eigenvalues for

$t \in [0,1]$ and z in a certain domain D of $\mathbb{R}^m$ (defined more precisely in chapter 3).
From the analysis of singular perturbations (see c.f. Hoppenstaedt (1971), O'Malley (1975),
Eckhaus (1979), Flaherty and O'Malley (1980) and Ringhofer (1981) we know that under these
assumptions we can expect a smooth solution in the interior of $[0,1]$ while at the
endpoints boundary layers will occur. Therefore the usual discretization methods are not
applicable unless we impose conditions on the mesh. (See c.f. Ascher and Weiss (1980) for
the behaviour of collocation methods of Gauss, Radau and Lobatto-type.) The standard
theory for discretization methods for general grids is not applicable unless the maximal
stepsize is smaller than ε. Flaherty and O'Malley (1980) avoid this difficulty by solving
only the reduced problem (which is independent of ε) instead of solving the full problem
(1.1): As ε tends to zero the solution (y,z) of (1.1) will converge to a function
$(\tilde{y}(t),\tilde{z}(t))$ uniformly in compact subsets of $(0,1)$ with a convergence rate
$O(\varepsilon)$. Obviously $(\tilde{y},\tilde{z})$ satisfies

(1.4) $\qquad\qquad\qquad 0 = f(t,\tilde{y},\tilde{z},0) \quad , \quad \tilde{z}' = g(t,\tilde{y},\tilde{z},0) \quad .$

To represent $(\tilde{y},\tilde{z})$ as the solution of a boundary-value problem (the reduced problem) we
need m boundary conditions for the equation (1.4). In the quasilinear case this can be
done by a "cancellation law" (see Flaherty and O'Malley (1980)). In compact subsets of
$(0,1)$ this gives the solution (y,z) up to the global discretization error we make by
approximating the reduced solution and a term of order $O(\varepsilon)$. Inside the layer this gives
us no approximation. Using some information about the analytical structure of the solution
we construct a method which is designed especially for problems of the type (1.1): It
shall solve (1.1) directly. So we need not compute the boundary conditions for the reduced
problem numerically. The amount of labour (i.e. of gridpoints) which is required to obtain
a uniform approximation of the solution on $[0,1]$ shall be "reasonable" (that means not
proportional to ε^{-1}). If we use a general (so for instance a uniform) mesh with maximal
stepsize $H >> \varepsilon$ the approximation shall be as good as the one obtained by solving the
reduced problem only. The solution will consist of a smooth part and two "layer-parts"
exponentially decaying from the endpoints. More precisely we have

$$y^-(t,\varepsilon) = y^-(t) + \mu(\tfrac{t}{\varepsilon}) + O(\varepsilon)$$

$$y^+(t,\varepsilon) = y^+(t) + \nu(\tfrac{1-t}{\varepsilon}) + O(\varepsilon)$$

$$z(t,\varepsilon) = z(t) + O(\varepsilon)$$

(y^- and y^+ are the first n_- and last n_+ components of y.) μ and ν decay exponentially as their arguments tend to infinity. $\mu(\tfrac{t}{\varepsilon})$ behaves roughly like the solution ζ of

(1.6) $$\varepsilon\,\zeta'(t,\varepsilon) = \lambda\,\zeta(t,\varepsilon), \quad \zeta(0,\varepsilon) = 1, \quad \mathrm{Re}\,\lambda < 0 \ .$$

A discretization-method which gives us a good approximation of y^- for an arbitrary mesh should also be able to deal with (1.6) on a uniform grid. We now compare the behaviour of the Box-scheme and the implicit Euler scheme at the problem (1.6). We have

$$\varepsilon\,\frac{\zeta_{i+1}^{(1)} - \zeta_i^{(1)}}{h} = \frac{\lambda}{2}\,(\zeta_i^{(1)} + \zeta_{i+1}^{(1)}), \quad \varepsilon\,\frac{\zeta_{i+1}^{(2)} - \zeta_i^{(2)}}{h} = \lambda\zeta_{i+1}^{(2)}$$

(1.7)

$$\zeta_i^{(1)} = (1 + \tfrac{\lambda h}{2\varepsilon})^i (1 - \tfrac{\lambda h}{2\varepsilon})^{-i} \sim (-1)^i, \quad \zeta_i^{(2)} = (1 - \tfrac{\lambda h}{\varepsilon})^{-i} \sim e^{\frac{i\lambda h}{\varepsilon}}$$

(1.7) shows that only the implicit Euler scheme gives us a good approximation of the solution. The reason for that is, that the growth function of the Box-scheme $(1+z)(1-z)^{-1}$ tends to -1 as z tends to infinity whereas the growth function of the implicit Euler scheme $(1-z)^{-1}$ tends to zero. This is the basic idea of a method developed by Kreiss and Nichols (1975): They use the implicit Euler scheme for y^-. Since the eigenvalues of f_1^+ are strictly unstable, they use the explicit Euler-scheme for y^+. For z they use the Box-scheme. We extend this idea to higher order methods and our approach for this extension is polynomial collocation. For a positive integer q we choose $q+1$ reference points $0 = u_0 < .. < u_q = 1$ on the interval $[0,1]$ according to

(1.8) $$u_j = \frac{j}{q}, \quad j = 0(1)q \ .$$

For a given mesh $T_H := \{0 = t_0 < .. < t_N = 1, \ H = \max(t_{i+1} - t_i)\}$ we choose the collocation points t_{ij} according to

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(1.9) $\qquad t_{ij} = t_i + h_i u_j \qquad h_i := t_{i+1} - t_i, \; j = 0(1)q, \; i = 0(1)N-1$.

Note that $t_{iq} = t_{i+1,0}$ holds. On the subinterval (t_i, t_{i+1}) we now use for y^- the collocation points $t_{i1}, \ldots, t_{iq}$. For y^+ we use $t_{i0}, \ldots, t_{i,q-1}$. For z we use $t_{i0}, \ldots, t_{iq}$. For $q = 1$ this gives the method of Kreiss and Nichols except that we use the trapezoidal rule for z.

We show that if the reduced problem has an isolated solution, the operator built by $F_{\varepsilon h}$ and the boundary conditions is stable in the sense of Keller (1975) and that the stability constant is independent of ε and the grid T_H. So if we use a mesh selecting strategy to achieve an over all accuracy δ we only have to control the local discretization error. We develop such a stepsize algorithm and show that the amount of gridpoints required to achieve an over-all-accuracy δ is essentially independent of ε and proportional to $\delta^{-\frac{1}{q}}$. This is comparable with the amount of gridpoints our method would need for an unperturbed problem. If we do not use a stepsize algorithm we can show the following result: For an arbitrary mesh we define the global error e_h by

$e_h = (y_h - y(t_h, \varepsilon), \; z_h - z(t_h, \varepsilon))$ where $(y(t_h, \varepsilon), \; z(t_h, \varepsilon))$ is the solution of the continuous problem restricted to the grid T_H. Then we have

$$
|e_{ij}| < c[\varepsilon H^q + H^{q+1} + \varepsilon + \exp(-\frac{\lambda t_i}{\varepsilon}) + \exp(\lambda \frac{t_{i+1}^{-1}}{\varepsilon})
$$

(1.10)
$$
+ \prod_{r=0}^{i-1} (1 + \frac{\kappa h_r}{\varepsilon})^{-1} + \prod_{r=i+1}^{N-1} (1 + \frac{\kappa h_r}{\varepsilon})^{-1} \,], \; j = 1(1)q, \; i = 0(1)N-1
$$

for some positive constants c, κ, λ (here H denotes the maximal stepsize). So away from the boundary we have the normal order of convergence $O(H^{q+1})$ of our method plus a term of order $O(\varepsilon)$. Thus the approximation is as good as if we had solved the reduced problem only.

This paper is organized as it follows. In chapter 2 we define the collocation scheme and introduce some basic notations. In chapter 3 we present some analytical results of singular perturbation theory. For the proofs we only refer to the literature. Furthermore

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we impose the main assumptions to be satisfied by the continuous problem. In chapter 4 we state our main results and explain them in a more detailed manner. In chapter 6 we prove them. In chapter 5 we demonstrate our method on a model problem where the continuous solution is known.

2. The method.

In this chapter we define the collocation method and introduce some basic notations which we will use throughout the rest of the paper.

(2.1) Definition: We call a set of $q+1$ points $0 \leqslant u_0 < u_1 < .. < u_{q-1} < u_q \leqslant 1$ a set of reference points in the unit interval.

(2.2) Definition: Given $N+1$ gridpoints $0 = t_0 < .. < t_N = 1$ and a set of $q+1$ reference points we define for each subinterval $[t_i, t_{i+1}]$ $q+1$ collocation points by

(2.3) $t_{ij} := t_i + h_i u_j \quad h_i := t_{i+1} - t_i, \quad j = 0(1)q, \quad i = 0(1)N-1$.

(Note that $t_{i0} = t_{i-1,q} = t_i$ holds.)

(2.4) Notation: We denote the mesh consisting of the gridpoints and the collocation points and its maximal stepsize by

$$T_H := \{t_{ij}, j = 0(1)q, i = 0(1)N-1\}, \quad H = \max_i h_i .$$

(2.5) Definition: For a given mesh T_H we define the scalar (vector-, matrix-) grid function x_h as a sequence of numbers (vectors, matrices) by

$$x_h := \{x_{ij}, j = 0(1)q, i = 0(1)N-1, x_{iq} = x_{i+1,0}\} .$$

(2.6) Definition: In the space of grid functions we introduce a norm by

$$\|x_h\|_h = \max\{\|x_{ij}\|, j = 0(1)q, i = 0(1)N-1\}$$

where $\|\cdot\|$ denotes the modulus or the max-norm in $\mathbb{R}^n$ or the matrixnorm introduced by the max norm.

(2.7) Notation: We denote the first n_- components (rows) and the last n_+ components (rows) of the vector (matrix) x with x^- and x^+ where n_- and n_+ are the dimensions of f_1^- and f_1^+ in (1.3).

(2.8) Definition: To apply a (polynomial) collocation method one generally seeks a polynomial spline function, which satisfies the given differential equation at certain points (the collocation points) as well as the boundary conditions. In order to approximate the solution $(y(= (y^-, y^+)^T), z)$ of (1.1) we construct a vector-spline function (p_y, p_z), $p_y = (p_y^-, p_y^+)^T$ which satisfies:

α) $p_y, p_z \in C_{[0,1]}$

β) p_y^- and p_y^+ are polynomials of degree q; p_z is a polynomial of degree $q+1$ in each subinterval $[t_i, t_{i+1}]$.

γ) To generalize the implicit and explicit Euler method we request that p_y^- satisfies the differential equation at all collocation points t_{ij} except the first one in each subinterval and p_y^+ satisfies the differential equations at all collocation points except the last one in each subinterval.

$$\epsilon(p_y^-)' = f^-(t, p_y, p_z, \epsilon) \quad \text{for} \quad t = t_{ij}, \ j = 1(1)q, \ i = 0(1)N-1$$

(2.9)

$$\epsilon(p_y^+)' = f^+(t, p_y, p_z, \epsilon) \quad \text{for} \quad t = t_{ij}, \ j = 0(1)q-1, \ i = 0(1)N-1 \ .$$

For z we take all $q+1$ collocation points except for 2 subintervals . For reasons which will be explained in chapter 6 it is necessary to modify the method used for z in order to obtain a higher order method. Given a mesh T_H we assume, that I_1 steps h_i on the left hand side and I_2 steps on the right hand side are of order $O(\epsilon)$ (where we also allow I_1 and I_2 to be zero):

$$h_i = O(\epsilon) \quad i = 0(1)I_1 - 1, \ i = I_2(1)N-1$$

$$h_{I_1} \gg \epsilon \quad h_{N-I_1-1} \gg \epsilon \ .$$

For the subinterval $[t_{I_1}, t_{I_1+1}]$ we take the collocation points $t_{I_1,1}, \ldots, t_{I_1,q}$ (as for p_y^-). For $[t_{N-I_2-1}, t_{N-I_2}]$ we take $t_{N-I_2-1,0}, \ldots, t_{N-I_2-1,q-1}$ so we have:

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$$p_z^1 = g(t, p_y, p_z, \epsilon) \quad \text{for} \quad t = t_{ij}, \qquad \begin{array}{l} j = 1(1)q \quad \text{if} \quad i = I_1 \\[4pt] j = 0(1)q-1 \quad \text{if} \quad i = N-I_2-1 \\[4pt] j = 0(1)q \quad \text{else} \ . \end{array}$$

For our analysis it will be convenient to rewrite (2.4) into a difference scheme. If we denote the values of p_y and p_z at $t = t_{ij}$ by y_{ij} and z_{ij} (2.4) can be written (see Weiss (1974)) as

(2.5)

$$\epsilon \frac{\bar{y}_{ij} - \bar{y}_{i0}}{h_i} = \sum_{k=1}^{q} w_{jk}^- \, f^-(t_{ik}, y_{ik}, z_{ik}, \epsilon)$$

$$\epsilon \frac{y_{ij}^+ - y_{i0}^+}{h_i} = \sum_{k=0}^{q-1} w_{jk}^+ \, f^+(t_{ik}, y_{ik}, z_{ik}, \epsilon)$$

$$\frac{z_{ij} - z_{i0}}{h_i} = \sum_{k=0}^{q} w_{jk}^0 \, g(t_{ik}, y_{ik}, z_{ik}, \epsilon)$$

$$j = 1(1)q, \ i = 1(1)N-1$$

$$w_{jk}^- = \int_0^{u_j} \frac{w(s) \, (u_k - u_0)}{(s-u_k) w'(u_k)(s-u_0)} \, ds, \quad w_{jk}^+ = \int_0^{u_j} \frac{w(s) \, (u_k - u_q)}{(s-u_k) w'(u_k)(s-u_q)}$$

$$w_{jk}^0 = \int_0^{u_j} \frac{w(s)}{w'(u_k)(s-u_k)} \, ds$$

$$w(s) := (s-u_0) \cdots (s-u_q) \ .$$

(2.6) Notation: We define the discrete operators D_h, S_h^-, S_h^+, S_h^0 by

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$$(D_h x_h)_{ij} = \frac{x_{ij} - x_{i0}}{h_i}$$

$$(S_h^- x_h)_{ij} = \sum_{j=1}^{q} w_{jk}^- x_{ik} \quad (S_h^+ x_h)_{ij} = \sum_{j=0}^{q-1} w_{jk}^+ x_{ik}$$

$$(S_h^0 x_h)_{ij} = \begin{cases} (S_h^- x_h)_{ij} & \text{if } i = I_1 \\[2mm] (S_h^+ x_h)_{ij} & \text{if } i = N - I_2 - 1 \end{cases}$$

$$\sum_{k=0}^{q} w_{jk}^0 x_{ik} \quad \text{otherwise} .$$

With this notation the discrete problem can be written as

$$(2.7) \qquad\qquad F_{\epsilon h}(y_h, z_h) = 0$$

$$\epsilon D_h y_h^- - S_h^- f^-(t_h, y_h, z_h, \epsilon)$$

$$F_{\epsilon h}(y_h, z_h) = \epsilon D_h y_h^+ - S_h^+ f^+(t_h, y_h, z_h, \epsilon)$$

$$D_h z_h - S_h^0 g(t_h, y_h, z_h, \epsilon) .$$

(2.8) Remark: To this point we did not restrict ourselves to a certain set of reference points. A natural choice would be Lobato-points in order to achieve the highest possible order of accuracy for z. However, for the rest of this paper we restrict ourselves to equidistant reference points.

3. Analytical results and main assumptions

Quasilinear singularly perturbed b.v.p.'s of the form (1.1) have been studied by several authors (see c.f. O'Malley (1974), Flaherty and O'Malley (1980), Howes (1980), Ringhofer (1981)). In this section we present their results in a manner which is appropriate for our purposes. One can find the proofs either directly in the above mentioned papers or can prove the results of this section analogously using the special structure of problem (1.1).

The solution (y,z) of (1.1) has a uniformly valid asymptotic expansion of the form

$$(3.1) \qquad y(t,\epsilon) \sim \sum_{j=0}^{\infty} \; (\tilde{y}_j(t) + \mu_j(\tfrac{t}{\epsilon}) + \nu_j(\tfrac{1-t}{\epsilon})) \epsilon^j$$

$$(3.2) \qquad z(t,\epsilon) \sim \sum_{j=0}^{\infty} \; \tilde{z}_j(t) \epsilon^j + \sum_{j=1}^{\infty} \; (\alpha_j(\tfrac{t}{\epsilon}) + \beta_j(\tfrac{1-t}{\epsilon})) \epsilon^j$$

where $\mu_j, \nu_j, \alpha_j, \beta_j$ are exponentially decaying as their arguments tend to infinity. (We call a function $\psi(\tau)$ exponentially decaying if it satisfies $|\psi(\tau)| \leq c_1 \exp(-c_2 \tau)$ for positive constants c_1 and c_2.) To derive this expansion it is necessary to define the *reduced problem*. This is done in Flaherty and O'Malley (1980) for the case of separated boundary conditions. In our case we proceed analogously.

$(\tilde{y}_0, \tilde{z}_0), \mu_0$ and ν_0 satisfy

$$(3.3) \qquad 0 = f(t, \tilde{y}_0, \tilde{z}_0, 0), \; \tilde{z}_0' = g(t, \tilde{y}_0, \tilde{z}_0, 0)$$

$$(3.4) \qquad \frac{d}{d\tau} \mu_0(\tau) = f_1(0, \tilde{z}_0(0)) \cdot \mu_0(\tau), \; \frac{d}{d\tau} \nu_0(\tau) = -f_1(1, \tilde{z}(1)) \nu_0(\tau)$$

$$(3.5) \qquad b_1(\tilde{z}_0(0), \tilde{z}_0(1)) \begin{pmatrix} \tilde{y}_0(0) + \mu_0(0) \\ \tilde{y}_0(1) + \nu_0(0) \end{pmatrix} + b_2(0, 0, \tilde{z}_0(0), \tilde{z}_0(1), 0) = 0 \; .$$

Since ν_0 and μ_0 must decay exponentially $\mu_0^+(0) = 0$, $\nu_0^-(0) = 0$ must hold. We split up b_1, into

$$(3.6) \qquad b_1(z_0, z_1) \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = b_{11}(z_0, z_1) \begin{pmatrix} y_0^- \\ y_1^+ \end{pmatrix} + b_{12}(z_0, z_1) \begin{pmatrix} y_0^+ \\ y_1^- \end{pmatrix} \; .$$

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(Here y^- and y^+ are defined as in (1.5).) So, b_{11} and b_{12} are $(n+m) \times n$ matrices. We assume that b_{11} has maximal rank in a domain of R^{2m} containing $(\tilde{z}_0(0), \tilde{z}_0(1))$ and that there exists a regular $(n+m) \times (n+m)$ matrix $W(z_0, z_1)$ such that

$$(3.7) \qquad W(z_0, z_1) b_{11}(z_0, z_1) = \begin{pmatrix} V(z_0, z_1) \\ 0 \end{pmatrix}$$

holds where V is a regular $n \times n$ matrix. In that case (3.5) is equivalent to

$$(3.8) \qquad W^+(\tilde{z}_0(0), \tilde{z}_0(1))(b_{12}(\tilde{z}_0(0), \tilde{z}_0(1))) \begin{pmatrix} \vec{y}_0^+(0) \\ y_-^-(1) \end{pmatrix} + b_2(0, 0, \tilde{z}_0(0), \tilde{z}_0(1), 0) = 0$$

$$(3.9) \qquad \begin{pmatrix} \mu_0^-(0) \\ \nu_0^+(0) \end{pmatrix} = -V^{-1}(\tilde{z}(0), \tilde{z}(1)) W^-(\tilde{z}(0), \tilde{z}(1)) b(\tilde{y}(0), \tilde{y}(1), \tilde{z}(0), \tilde{z}(1), 0) \ .$$

(Here W^- and W^+ denote the first n_- and the last n_+ rows of W.) (3.8) together with (3.3) gives the reduced problem. If the reduced problem has an isolated solution we can show the uniform validity of the asymptotic expansion (3.1), (3.2). This leads to the following assumptions: There exists a domain D of R^m so that

(3.10) H1: $f_1(t, z)$ is a block diagonal matrix. So

$$f_1(t, z) = \begin{pmatrix} f_1^-(t, z) & 0 \\ 0 & f_1^+(t, z) \end{pmatrix}$$

holds. There exist positive constants $\bar{\lambda}_-$ and $\bar{\lambda}_+$ so that all eigenvalues λ of the n_- dimensional block $f_1^-(t, z)$ satisfy Re $\lambda < -\bar{\lambda}$ and all eigenvalues of the n_+ dimensional block $f_1^+(t, z)$ satisfy Re $\lambda > \bar{\lambda}_+$ for all $t \in [0,1]$ and $z \in D$.

(3.11) H2: There exist matrices $W(z_0, z_1)$ and $V(z_0, z_1)$ as defined in (3.7) for all $z_0, z_1 \in D$.

(3.12) H3: The reduced problem (3.3), (3.8) has an isolated solution $(\tilde{y}_0(t), \tilde{z}_0(t))$. $\tilde{z}_0(t)$ lies within a compact subset of D for all $t \in [0,1]$.

(3.13) H4: f, g, and b are as often differentiable as necessary with respect to all of their arguments.

Under these assumptions the asymptotic expansion (3.1), (3.2) is uniformly valid.

(3.14) Notation: For the further we denote with $(Wb)(y,z) = 0$ the boundary condition.

$$W(z(0,\varepsilon),\ z(1,\varepsilon))b(y(0,\varepsilon),y(1,\varepsilon),z(0,\varepsilon),z(1,\varepsilon)) = 0 \quad .$$

4. Main results

We first show, that under the assumptions (3.10)-(3.13) there exists also a solution of the discrete problem

$$(4.1) \qquad\qquad F_{\varepsilon h}(y_h,z_h) = 0, \quad b_\varepsilon(y_h,z_h) = 0$$

and this solution is stable in the sense of Keller (1975). We start with constructing a uniform $O(\varepsilon+H)$ approximation of (y_h,z_h): We define the discrete reduced problem by

$$(4.2) \qquad\qquad F_h^R(y_h,z_h) = 0, \quad b^R(y_h,z_h) = 0$$

$$(F_h^R)_{ij} = \begin{pmatrix} f(t_{ij},y_{ij},z_{ij},0) \\ D_{ij}z_h - S_{ij}^0[g(t_h,y_h,z_h,0)] \end{pmatrix} \quad j = 1(1)q,\ i = 0(1)N-1$$

$$b^R(y_h,z_h) = W_2(z_{00},z_{N0})b(y_{00},y_{N0},z_{00},z_{N0},0) \quad .$$

Because of hypothesis (3.12), (4.2) has an isolated solution $(\tilde{y}_h,\tilde{z}_h)$ (see Keller (1975)). We now define $\hat{y}_h, y_h^0, z_h^0$ by

$$\varepsilon D_h\hat{y}_h^- = S_h^-(f_1(t_h,\tilde{z}_h)\hat{y}_h^-), \quad \varepsilon D_h\hat{y}_h^+ = S_h^+(f_1^+(t_h,\tilde{z}_h)\hat{y}_h^+)$$

$$(4.3) \qquad \begin{pmatrix} \hat{y}_{00}^- \\ \hat{y}_{N0}^+ \end{pmatrix} = -v(\tilde{z}_{00},\tilde{z}_{N0})^{-1} W_1(\tilde{z}_{00},\tilde{z}_{N0})b(\tilde{y}_{00},\tilde{y}_{N0},\tilde{z}_{00},\tilde{z}_{N0},0)$$

$$y_h^0 = \tilde{y}_h + \hat{y}_h, \quad z_h^0 = \tilde{z}_h \quad .$$

(y_h^0,z_h^0) is a uniform $O(\varepsilon)$ approximation of the solution (y_h,z_h) of (1.18). To show the existence and the stability of (y_h,z_h) we need that the linearization of the operator built by $F_{\varepsilon h}$ together with the boundary conditions (3.14) at (y_h^0,z_h^0) has an inverse bounded uniformly in ε and H. This gives a linear difference operator $(L_{\varepsilon h},B)$ of the

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form

$$(4.3) \qquad L_{\epsilon h}(y_h, z_h) = \begin{pmatrix} \epsilon \, D_h y_h^- - S_h^- [(A_h^{11})^- y_h^- + (A_h^{12})^- z_h] \\[6pt] \epsilon \, D_h y_h^+ - S_h^+ [(A_h^{11})^+ y_h^+ + (A_h^{12})^+ z_h] \\[6pt] D_h z_h - S_h^0 [A_h^{21} y_h + A_h^{22} z_h] \end{pmatrix}$$

$$(4.4) \qquad B(y_h, z_h) = \begin{pmatrix} B^{11} \cdot (y_{00}^-, y_{N0}^+)^T + B^{12} (y_{00}^+, y_{N0}^-, z_{00}, z_{N0})^T \\[6pt] B^2 (y_{00}^+, y_{N0}^-, z_{00}, z_{N0}) \end{pmatrix} .$$

Since y_h^0 will exhibit a rapid transition at the endpoints the matrices A_h^{ij} will consist of a smooth part and a part exponentially decaying away from the boundary i.e. they will satisfy

$$A_h^{k1} = \tilde{A}_h^{k1} + \hat{A}_h^{k1}, \quad |D_h \tilde{A}_h^{k1}|_h \le c_1, k, 1 = 1, 2, \quad \hat{A}_h^{11} = 0$$

$$|\hat{A}_{ij}^{21}| \le c_2 \prod_{r=0}^{i-1} (1 + \frac{\sigma h_r}{\epsilon})^{-1} + \prod_{r=i+1}^{N-1} (1 + \frac{\sigma h_r}{\epsilon})^{-1}, \quad 1 = 1, 2$$

$$(4.5)$$

$$|(A_{ij}^{12})^-| \le c_2 \prod_{r=0}^{i-1} (1 + \frac{\gamma h_r}{\epsilon})^{-1}, \quad |(A_{ij}^{12})^+| \le \prod_{r=i+1}^{N-1} (1 + \frac{\gamma h_r}{\epsilon})^{-1}$$

$$j = 0(1)q, \quad i = 0(1)N-1 \ .$$

Because B is the linearization of the boundary conditions (3.14) B will satisfy $\det(B^{11}) \ne 0$.

For the problem

$$(4.6) \qquad L_{\epsilon h}(y_h, z_h) = \phi_h, \quad B(y_h, z_h) = \beta$$

(where ϕ_h is a vector grid function of appropriate dimension and $\beta \in R^{n+m}$) we define the reduced problem by

$$(4.7) \qquad L_h^R(y_h, z_h) = \phi_h, \quad B^R(y_h, z_h) = \beta_2$$

$$(4.8) \qquad L_h^R(y_h, z_h) = \begin{cases} -S_h^-[(A_h^{11})^- y_h^- + (A_h^{12})^- z_h] \\ -S_h^+[(A_h^{11})^+ y_h^+ + (A_h^{12})^+ z_h] \\ D_h z_h - S_h^0[A_h^{21} y_h - A_h^{22} z_h] \end{cases}$$

$$(4.9) \qquad B^R(y_h, z_h) = B^2(y_{00}^+, y_{N0}^-, z_{00}, z_{N0}) \ .$$

(β_2 denotes the last m components of β). For the so defined linear operator we have the following stability result

(4.10) **Theorem:** Let $L_{\varepsilon h}, B, L_h^R, B^R$ be defined as in (4.3), (4.4), (4.8), (4.9). If there exists a constant c_1, so that L_h^R satisfies

$$(4.11) \qquad \|(y_h, z_h)\|_h \leq c_1 (\|L_h^R(y_h, z_h)\|_h + \|B^R(y_h, z_h)\|)$$

then the problem

$$(4.12) \qquad L_{\varepsilon h}(y_h, z_h) = \phi_h, \quad B(y_h, z_h) = \beta$$

has a solution (y_h, z_h) for all vector-grid functions ϕ_h and all vectors β. There exists a constant c_2 independent of ε so that (y_h, z_h) satisfies:

$$(4.13) \qquad \|(y_h, z_h)\|_h \leq c_2 (\|\phi_h\|_h + \|\beta\|) \ .$$

Since the continuous reduced problem has an isolated solution the linearization of

$$(4.14) \qquad F_{\varepsilon h}(y_h, z_h), \quad (Wb)(y_h, z_h)$$

satisfies the hypotheses of theorem (4.10). (Here $(Wb)(y_h, z_h)$ is defined analogously to (3.14).) Using theorem (4.10) we can show the existence of a solution of (4.1). Moreover the operator defined by $F_{\varepsilon h}$ and the boundary conditions Wb is stable in the sense of Keller (1975) in a neighbourhood of (y_h, z_h) of (4.1). The neighbourhood and the stability constant are independent of ε. So we have

(4.15) **Theorem:** If the continuous problem (1.1) satisfies the hypotheses in chapter 2 then there exist some positive constants ε_0, H_0 so that (4.1) is solvable for all $\varepsilon < \varepsilon_0$ and $H < H_0$. The solution (y_h, z_h) of (4.1) satisfies

$$(4.16) \qquad \|(y_h, z_h) - (y_h^0, z_h^0)\|_h \leq c_3 \varepsilon$$

for some positive constant c_3 independent of ε. Here (y_h^0, z_h^0) are defined as in

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(4.3). The stability-inequality

$$\|(y_h^1, z_h^1) - (y_h^2, z_h^2)\|_h \le c_4 \|F_{ch}(y_h^1, z_h^1) - F_{ch}(y_h^2, z_h^2)\|_h +$$

(4.17)

$$+ \|(Wb)(y_h^1, z_h^1) - (Wb)(y_h^2, z_h^2)\|$$

is satisfied for all (y_h^i, z_h^i), $i = 1,2$ satisfying $\|(y_h^i, z_h^i) - (y_h^0, z_h^0)\|_h \le p$ and all $H < H_0$, $\epsilon < \epsilon_0$.

Since (y_h^0, z_h^0) in (4.3) is an $O(\epsilon)$ approximation of (y_h, z_h) the global error $((y_h, z_h) - (y(t_h, \epsilon), z(t_h, \epsilon)))$ is up to terms of order $O(\epsilon)$ given by the difference between (y_h^0, z_h^0) and the $O(1)$ terms in the asymptotic expansion (2.1), (2.2) of $(y(t,\epsilon), z(t,\epsilon))$:

(4.18) Theorem: Let (y_0, z_0) denote the $O(1)$ term in the asymptotic expansion of $(y(t,\epsilon), z(t,\epsilon))$. Then there exist positive constants c and κ so that

$$\|(y_0(t_{ij}, \epsilon), z_0(t_{ij}, \epsilon)) - (y_{ij}^0, z_{ij}^0)\| \le c(H^{q+1} +$$

$$\prod_{r=0}^{i-1} (1 + \frac{\kappa h_r}{\epsilon})^{-1} + \prod_{r=i+1}^{N-1} (1 + \frac{\kappa h_r}{\epsilon})^{-1} + \exp(-\frac{\gamma_- t_{ij}}{\epsilon}) +$$

$$\exp(\gamma_+ \frac{t_{ij} - 1}{\epsilon}))$$

holds for all $H < H_0$ and $\epsilon < \epsilon_0$ where γ_- and γ_+ are the (in modulus) smallest real parts of $f_1^-(0, \tilde{z}(0))$ and $f_1^+(1, \tilde{z}(1))$.

From Theorem (4.18) follows immediately:

(4.20) Theorem: Let $(y(t,\epsilon), z(t,\epsilon))$ and (y_h, z_h) be the solutions of (1.1) and (3.1). Then there exist positive constants c_6, γ and κ so that

$$\|(y(t_{ij}), z(t_{ij})) - (y_{ij}, z_{ij})\| \le c_6(H^{q+1} + \epsilon +$$

$$\prod_{r=0}^{i-1} (1 + \frac{\kappa h_r}{\epsilon})^{-1} + \prod_{r=i+1}^{N-1} (1 + \frac{\kappa h_r}{\epsilon})^{-1} + \exp(\frac{-\gamma t_{ij}}{\epsilon}) +$$

$$\exp(\lambda_+ \frac{t_{ij} - 1}{\epsilon}))$$

holds for all $\epsilon < \epsilon_0$ and all grids T_H with $H < H_0$.

Away from the boundary we will have an approximation of order $O(H^{q+1}+\epsilon)$ for an arbitrary (so for instance also uniform) mesh. So in this case we have the order of convergence which we would obtain by solving the reduced problem plus a term of order $O(\epsilon)$. In that sense our method is equivalent to solving only the reduced problem except that we need not compute the boundary conditions of the reduced problem. We now consider the case where we want a better approximation than the one we obtain by using an arbitrary mesh. This is of some interest if either ϵ is rather big (let's say $\epsilon = 10^{-2}$) or if we are interested in the solution inside the layers.

If we can show now that $(y(t_h,\epsilon),z(t_h,\epsilon))$ (the restriction of the exact solution $(y(t,\epsilon),z(t,\epsilon))$ on the mesh T_H) lies within the ball $K_\rho(y_h^0,z_h^0)$, theorem (4.15) tells us that we only have to control the local discretization error to obtain a uniform approximation of the solution. In other words if we choose the mesh T_H so that

$$\|F_{\epsilon h}(y(t_h,\epsilon),\ z(t_h,\epsilon))\|_h < \delta$$

holds we have $\|(y_h,z_h) - (y(t_h,\epsilon),z(t_h,\epsilon))\| = O(\delta)$.

(4.22) **Lemma:** Let the local discretization error (ℓ_h^y,ℓ_h^z) be defined by

(4.22) $$(\ell_h^y,\ell_h^z) = F_{\epsilon,h}(y(t_h,\epsilon),z(t_h,\epsilon))$$

where $(y(t_h,\epsilon),z(t_h,\epsilon))$ denotes the restriction of the solution of (1.1) to the grid T_H. Then there exists a positive constant c_7 such that

$$\|(\ell_{ij}^y,\ell_{ij}^z)\| \leq c_7(ch_i^q + h_i^{q+1} + (\frac{h_i}{\epsilon})^q \exp(-\frac{\lambda_- \bar{t}_i}{q\epsilon})$$

(4.23) $$+ (\frac{h_i}{\epsilon})^{q+1} \exp(-\frac{\lambda_- \bar{t}_i}{(q+1)\epsilon} + (\frac{h_i}{\epsilon})^q \exp(\lambda_+ \frac{\bar{t}_i - 1}{q\epsilon}) + (\frac{h_i}{\epsilon})^{q+1}$$

$$\exp(\lambda_+ \frac{\bar{t}_i - 1}{(q+1)\epsilon}))$$

holds for $j = 0(1)q$, $i = 0(1)N-1$ and some $\bar{t}_i \epsilon[t_i,t_{i+1}]$.

So if we want to guarantee that $\|(\ell_h^y,\ell_h^z)\|_h = O(\delta)$ holds for a certain desired accuracy δ we have to choose the mesh T_H according to

$$h_i \leq \min\{\delta^{\frac{1}{q}} \epsilon^{-\frac{1}{q}}, \delta^{\frac{1}{q+1}}, \delta^{\frac{1}{q}} \epsilon \exp(\frac{\lambda_- t_i}{q\epsilon}), \delta^{\frac{1}{q+1}} \epsilon$$

(4.24)

$$\exp \frac{\lambda_- t_i}{(q+1)\epsilon}, \delta^{\frac{1}{q}} \epsilon \exp(\frac{-\lambda_+(\bar{t}_i - 1)}{q\epsilon}), \delta^{\frac{1}{q+1}} \epsilon \exp(\frac{-\lambda_+(\bar{t}_i - 1)}{(q+1)\epsilon})\} .$$

If we choose T_H according to (3.24) we can show that $\|(y_h, z_h) - (y(t_h, \epsilon), z(t_h, \epsilon)\| =$ $O(\epsilon+\delta)$ holds. Therefore (y, z) lies within the ball with radius ρ and center (y_h^0, z_h^0) where $F_{\epsilon h}$ is stable (see Theorem (4.15)) if only ϵ and δ are sufficiently small. Applying Theorem (4.15) gives

(4.25) Theorem: If T_H is chosen according to (4.24) then there exist positive constants c_8, ϵ_0 and δ_0 so that

(4.26) $$\|(y_h, z_h) - (y(t_h, \epsilon), z(t_h, \epsilon))\|_h < c_8 \delta$$

holds for any $\delta < \delta_0$ and $\epsilon < \epsilon_0$.

To construct the mesh T_H we proceed as it follows:

Step 1: Starting with h_0 we choose h_i according to $h_i \leq \min \{\epsilon\delta^{\frac{1}{q}} \exp(\frac{\lambda_- t_i}{q\epsilon})$,

$\epsilon\delta^{\frac{1}{q+1}} \exp \frac{\lambda_- t_i}{(q+1)\epsilon}\}$ as long as $h_i < \delta^{\frac{1}{q+1}}$ holds. (For a reasonable $\delta > \epsilon^{q+1}$

$\delta^{\frac{1}{q+1}} < \delta^{\frac{1}{q}} \epsilon^{\frac{1}{q}}$ holds.)

Step 2: Starting with h_{N-1} we choose h_i according

$$h_i \leq \min\{\epsilon\delta^{\frac{1}{q}} \exp \frac{-\lambda_+(t_{i+1} - 1)}{q\epsilon}, \epsilon\delta^{\frac{1}{q+1}} \exp \frac{-\lambda_+(t_{i+1} - 1)}{(q+1)\epsilon}\}$$

as long as $h_i < \delta^{\frac{1}{q+1}}$ holds.

Step 3: For the rest of the interval choose $h_i < \delta^{\frac{1}{q+1}}$.

Therefore it is necessary to know the constant λ_- and λ_+ which are the smallest moduli of the real parts of $f_1^-(0, \tilde{z}(0))$ and $f_1^+(1, \tilde{z}(1))$. These can either be computed by

solving the problem on a uniform mesh first or estimates for them can be obtained, if one uses a continuation method.

An immediately arising question is: How big is the additional amount of labour inside the layers? That means: How many grid-points do we need in steps 1 and 2? What one would like to have is an estimate for the number of grid-points, which is independent of ϵ. Unfortunately the number of grid-points required to achieve a certain accuracy δ tends to infinity as ϵ goes to zero, but the growth is so slow that for reasonable ranges of ϵ and δ it appears to be independent of ϵ. More precisely we have

(4.27) Lemma: Let $N(\delta,\lambda,y,\epsilon)$ denote the number of grid-points produced by step 1 (or 2) (so λ either equals λ_- or λ_+). Then we have

$$(4.28) \qquad |N(\delta,\lambda,q,\epsilon_1) - N(\delta,\lambda,q,\epsilon_2)| < 3$$

for δ,λ,q and ϵ_1 and ϵ_2 satisfying

$$\frac{\lambda\delta^{\frac{1}{q}}}{q} > 10^{-3}, \quad 10^{-18} < \epsilon_i < 10^{-2} \quad , \quad i = 1,2 \quad .$$

Furthermore

$$(4.29) \qquad N(\delta,\lambda,q,\epsilon) < \text{const } \frac{q}{\lambda} \delta^{-\frac{1}{q}} \ell n(\ell n\ (\frac{1}{\epsilon}))$$

holds.

(4.30) Remark: The estimate (4.29) is not sharp, but together with (4.28) it shows, that for reasonable choices of ϵ and δ the amount of additionally needed grid-points is of the same order or magnitude as a q-th order collocation method would need for on unperturbed problem.

5. Numerical Example

We now illustrate our results with a third order quasilinear model problem where we know the exact solution. For the computations a modified version of the packages LOBATO and SOLVEBLOCK (see de Boor-Weiss) (1976)) has been used. We consider the system

$$\epsilon y_1' = -(z - \epsilon y_1 + \epsilon^2 y_2 + \epsilon e^t - 5\epsilon^2 e^t)(y_1 - \epsilon y_2 - e^t + 5\epsilon e^t(y_2 - 5e^t + 1))$$

(5.1)
$$\epsilon y_2' = (z - \epsilon y_1 + \epsilon^2 y_2 + \epsilon e^t - 5\epsilon^2 e^t)(y_2 - 5e^t) + 5\epsilon e^t$$

$$z' = z - (\epsilon + e^t)(y_1 - \epsilon y_2 - e^t + 5\epsilon e^t)$$

$$y_1(0,\epsilon) + y_2(0,\epsilon) + z(0,\epsilon) - 8 - \epsilon = 0$$

(5.2)
$$y_1(1,\epsilon) + y_2(1,\epsilon) + z(1,\epsilon) - 7e - 1 - \epsilon = 0$$

$$y_1(1,\epsilon) + z(1,\epsilon) - 2e - \epsilon = 0 \quad .$$

Here the matrix $f_1(t,z)$ is of the form

(5.3)
$$f_1(t,z) = \begin{pmatrix} -z & 0 \\ 0 & z \end{pmatrix}$$

and we have the reduced problem

(5.4)
$$0 = \tilde{z}\tilde{y}_1 + \tilde{z}e^t, \; 0 = \tilde{z}\tilde{y}_2 - 5\tilde{z}e^t, \tilde{z}' = \tilde{z} - e^t(\tilde{y}_1 - e^t)$$

(5.5)
$$\tilde{z}(1) + \tilde{y}_1(1) = 2e$$

which has the solution $y_1 = e^t, y_2 = 5e^t, z = e^t$.

(5.1), (5.2) has the solution

$$y_1 = e^t + \exp(\frac{1-e^t}{\epsilon}) + \epsilon \exp(\frac{e^t-e}{\epsilon})$$

(5.6)
$$y_2 = 5e^t + \exp(\frac{e^t-e}{\epsilon})$$

$$z = e^t + \epsilon \exp(\frac{1-e^t}{\epsilon}) \quad .$$

We solved (5.1), (5.2) using the mesh selecting strategy (4.8). We varied the desired accuracy δ by halving $\delta^{\frac{1}{q+1}}$ starting with $\delta^{\frac{1}{q}} = \frac{1}{2}$. Thus the maximal error should decrease like $(\frac{1}{2})^q$. We varied ϵ from 10^{-2} to 10^{-10} in steps of 10^{-2}. By varying ϵ for fixed δ and q the norm of the errors differed by less than 1%:

Maximal errors of:

$\frac{1}{\delta^q}$	1	2	3	4
.25	.11	.81E-2	.64E-3	.32E-4
.13	.63E-1	.21E-2	.81E-4	.29E-5
.63E-1	.35E-1	.54E-3	.1 E-4	.12E-6
.18E-1	.14E-3	.13E-5	.13E-5	.73E-8

Note that the error-constants are decreasing for increasing q. (So for instance for
q = 4 the error constant is $0(10^{-2})$). To achieve the accuracy the following amounts of
meshpoints have been required:

Required meshpoints:

$\frac{1}{\delta^q}$ $\diagdown$ q	1	2	3	4
.25	17	25	33	41
.13	26	43	58	74
.63E-1	44	77	109	140
.31E-1	78	144	208	272

So for a fixed q the number of required meshpoints grows like $\frac{1}{\delta^q}$. Varying ϵ from
10^{-2} to 10^{-6} the amount of grid-points for fixed q and δ varyied only by $\pm$ 1.

Now we solve (4.1), (4.2) using a uniform mesh (h_i = H, i = 0(1)M-1). Because of the
theory in the preceding chapter we expect an error of order $(H^{q+1}+\epsilon)$ away from the
boundary layers. In the next table the norm of the error at t = 0.5 is listed for
q = 3.

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ε \ H	.1	.5E-1	.25E-1
.1E-2	.58E-3	.55E-3	.51E-3
.1E-4	.53E-5	.6E-5	.6E-5
.1E-6	.5E-6	.4E-7	.57E-7

So we see that the error is of order ε independent of H except for ε = .1E-6 and H = .1 where $cH^{q+1} > \varepsilon$ holds. (c is the error constant of our method for $q = 3$.)

6. Proofs:

To prove Theorem (4.10) we first need some auxiliary results. We start with scalar constant coefficient problems and estimate the growth function of the method we use for the component y^- in (1.18):

6.1 Lemma: Given the complex numbers x_0 and z and the vector $e = (e_1,\ldots,e_q)$ the solution $x = (x_1,\ldots,x_q)$ of the system

(6.2)
$$x_j = x_0 + z \sum_{k=1}^{q} w^-_{jk}x_k + e_j \quad , \quad j = 1(1)q$$

satisfies
$$x_j = \gamma_j(z)x_0 + \beta_j^T(z)e \quad .$$

β_j are vectors of dimension q. γ_j and the components of β_j are rational functions of z. For $q = 1(1)10$ there exist angles θ_q satisfying $0 < \theta_q < \frac{\pi}{2}$ and positive constants c_q such that

(6.3)
$$|\gamma_j(z)|, \; |\beta_j(z)|, \; |z\gamma_j(z)|, \; |z\beta_j(z)| < c_q$$

holds for z in $S_q = \{z | \mathrm{Re}\, z < 0, \; |\mathrm{Im}\, z(\mathrm{Re}\, z)^{-1}| < \tan \theta_q\}$ and $j = 1(1)q$. γ_q satisfies $|\gamma_q| < 1$ in S_q and $\gamma_q(0) = 1$.

Proof: (5.2) can be written in the form

$$x = \vec{1}\, x_0 + z\Omega^- x + e$$

where $\vec{1}$ denotes the vector $(1,\ldots,1)^T$ and Ω^- the $q \times q$ matrix $(w^-_{ij}, i,j=1,q)$. If

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we solve this system by Cramer's rule and look at the involved determinants as functions of z we can write the solution $(x_1,...,x_q)^T$ in the form (6.2). The γ_j and β_j turn out to be rational functions where the degree of the denumerator-polynomial is $q-1$ and the degree of the denominator polynomial is q. Therefore they behave like z^{-1} at $z = \infty$.

The growth functions γ_q have been analyzed numerically by Ueberhuber (1979) and are less then 1 in modulus within the sector S_q.

(6.15) Lemma: Let all eigenvalues λ of the $n \times n$ matrices A_h lie within a compact subset of the interior of S_q and satisfy $\mathrm{Re}\,\lambda < -\tilde{\lambda} < 0$ for a positive constant $\tilde{\lambda}$. Let $|D_h A_h|_h < c_{11}$ hold for a positive constant c_{11}. Then the solution y_h of

(6.16)
$$\varepsilon D_h y_h - S_h^-(A_h y_h) = f_h \quad, \quad y_{00} = \alpha \quad \text{satisfies}$$
$$|y_h|_h < c_{12}(|\alpha| + |f_h|_h)$$

for some positive constant c_{12} independent of ε. (Here y_h and f_h are vector grid functions of dimension n.)

Proof: We rewrite (6.16) in
$$\varepsilon(D_h y_h)_{ij} = A_{i0}(S_h^- y_h)_{ij} + h_i(S_h^-((D_h A_h)y_h))_{ij} + f_{ij} \quad j = 1(1)q, \quad i = 0(1)N-1 \quad.$$
Since $D_h A_h$ is bounded we get, by using Lemma (6.1)
$$y_{i+1,0} = \gamma_q(\frac{h_i}{\varepsilon} A_{i0})y_{i0} + \sum_{k=1}^{q} (\beta_{qk}(\frac{h_i}{\varepsilon} A_{i0}) \frac{h_i}{\varepsilon} f_{ik}) + O(h_i|y_h|_h) \quad.$$

Here the rational functions γ_q and β_q are applied to the nonsingular matrix $\frac{h_i}{\varepsilon} A_{i0}$ in the usual manner i.e.
$$\gamma_q(\frac{h_i}{\varepsilon} A_{i0}) = \int_\Gamma \gamma_q(\frac{h_i}{\varepsilon} \lambda)(\lambda I - A_{i0})^{-1} d\lambda$$

where Γ is a closed curve in the complex domain and all eigenvalues of A_{i0} lie within Γ. (See c.f. Dunford Schwartz (1957).)

$$|y_{10}| < |\alpha| + \text{const}\left(\sum_{j=0}^{M} \prod_{r=j}^{M} \left(1 - \frac{\kappa\gamma_r h_r}{\epsilon}\right)^{-1} \frac{h_j}{\epsilon} + \right.$$

(6.17)

$$\left. + \sum_{j=M+1}^{i-1} \prod_{r=j+1}^{i-1} \left(1 - \frac{\kappa\gamma_r h_r}{\epsilon}\right)^{-1}\right)(|f_h|_h + O(H|y_h|_h))$$

by estimating $\left(1 - \frac{\kappa\gamma_r h_r}{\epsilon}\right)^{-1} < \exp(\frac{-\kappa\tilde{\gamma}h_r w}{\epsilon})$ with some suitable constant w we get

(6.18) $\displaystyle\sum_{j=0}^{M} \sum_{r=j}^{M} \left(1 - \frac{\kappa\gamma_r h_r}{\epsilon}\right)^{-1} \frac{h_j}{\epsilon} < \sum_{j=0}^{M} \exp(\frac{-\kappa\tilde{\gamma}h_r}{\epsilon}) w(t_M - t_j)) \frac{h_j}{\epsilon} < \frac{1}{\epsilon}\int_0^{t_M} \exp(-\kappa w\tilde{\gamma}(\frac{t_r - s}{\epsilon}))ds$.

Since the integral is bounded and the other sum in (6.17) is geometric we get the desired estimate

(6.19) Lemma: Let A_h be a $n \times n$ matrix and $\tilde{f}_h$ and $\hat{f}_h$ be n-dimensional vector grid functions. Let A_h satisfy the conditions for the matrices A_h in Lemma (6.15). We assume that

$$|D_h A_h|_h < c_{13}, \quad |D_h \tilde{f}_h|_h < c_{13}$$

(6.20)

$$|\hat{f}_{ij}| < c_{14} \prod_{r=0}^{i} \left(1 + \frac{\kappa h_r}{\epsilon}\right)^{-1} \quad j = 1(1)q, \; i = 0,1,2,\ldots$$

holds for some positive constants c_{13}, c_{14} and κ. Then the solution y_h of

(6.21) $\qquad \epsilon D_h y_h - S_h^-(A_h y_h + \tilde{f}_h + \hat{f}) = 0, \; y_{00} = \alpha$ satisfies

$$y_h = -(A^{-1}\tilde{f})_h + \hat{y}_h + O(\epsilon), \quad |\hat{y}_{ij}| < c_{15} \prod_{r=0}^{i} \left(1 + \frac{\sigma h_r}{\epsilon}\right)^{-1}$$

(6.22)

$$j = 1(1)q, \; i = 0(1)N-1$$

for some positive constants c_{15} and σ where $\sigma < \min(\kappa, c_{12}^{-1})$ holds. (Here c_{12} is defined as in Lemma (6.15).

Proof: We define $\hat{y}_h$ and u_h by

(6.23) $\qquad \epsilon D_h \hat{y}_h = S_h^-(A_h \hat{y}_h + \hat{f}_h), \; j = 1(1)q, \; i = 0,1,\ldots \; \hat{y}_{00} = \alpha + A_{00}^{-1} f_{00}$,

(6.24) $\qquad u_h = y_h + (A^{-1}\tilde{f})_h - \hat{y}_h$, then u_h satisfies

(6.25) $\qquad \epsilon D_h u_h = S_h(A_h u_h) + \epsilon D_h(A_h^{-1} f_h)$.

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Because of Lemma (6.15) we obtain that $|u_h|_h = 0(\epsilon)$ holds. We define v_h by

$$(6.26) \qquad v_{ij} = \hat{y}_{ij} \prod_{r=0}^{i} (1 + \frac{\sigma h_r}{\epsilon}), \; j = 1(1)q, \; i = 0,1,\dots, v_{00} = \hat{y}_{00} \; ,$$

for some positive σ less than $\min(\kappa, c_6^{-1})$. v_h satisfies

$$(6.27) \quad \epsilon D_{ij} v_h = \sigma v_{i0} + S_{ij}^{-}(A_h v_h + p_h \hat{z}_h), \; p_{ij} = \prod_{r=0}^{i} (1 + \frac{\sigma h_r}{\epsilon}) \; j = 1(1)q, \; i = 0,1,\dots \; .$$

Because of (6.20) $|p_h \hat{z}_h|_h < c_{14}$ holds. So v_h is bounded and therefore (6.22) holds.

<u>(6.28) Lemma</u>: Let the operator associated with the problem

$$(6.29) \qquad D_h z_h = S_h^0(C_h z_h + \gamma_h), \; H(z_{00}, z_{N0}) = \eta$$

have a uniformly bounded inverse i.e. let $|z_h|_h < c_{16}(|\eta| + |\gamma_h|_h)$ hold for some

constant c_{16}. (Here z_h and γ_h denote vector grid functions of dimension m. H, η

and C_h are of appropriate dimensions.) Let the m-dimensional vectors f_h satisfy

$$|f_{ij}| < c_{17}(|e_{ij}^{-}| + |e_{ij}^{+}|), \; j = 0(1)q, \; i = 0(1)N$$

$$(6.30) \qquad e_{ij}^{-} = \prod_{r=0}^{i} (1 + \frac{c_{18} h_r}{\epsilon})^{-1}, \; j = 1(1)q, \; e_{ij}^{+} = \prod_{r=i}^{N-1} (1 + \frac{c_{18} h_r}{\epsilon})^{-1} \; ,$$

$$j = 0(1)q-1, \; i = 0(1)N-1$$

for some positive constants c_{17} and c_{18}. Then the solution z_h of

$$(6.31) \qquad D_h z_h - S_h^0(C_h z_h + f_h) = 0, \; H(z_{00}, z_{N0}) = \epsilon \beta \quad \text{satisfies}$$

$$(6.32) \qquad |z_h|_h < c_{19} \epsilon(|\beta| + c_{20})$$

for some positive c_{19} independent of ϵ.

<u>(6.36) Lemma</u>: Let $L_{\epsilon h}$ be defined as in (3.3). Let the coefficient matrices A_h^{ij}

satisfy (3.5). Assume that the operator associated with the reduced problem (3.7) has a

uniformly bounded inverse, i.e. that there exists a constant c_{22} so that

$$(6.37) \qquad |(y_h, z_h)|_h < c_{22}(|L_h^R(y_h, z_h)|_h + |B^R(y_h, z_h)|)$$

holds for all grids T_H. Then there exists a solution (y_h, z_h) of the problem

$$(6.38) \qquad L_{\epsilon h}(y_h, z_h) = \phi_h, \; y_{00}^{-} = \beta_0, \; y_{N0}^{+} = \beta_1, \; B^R(y_h, z_h) = \beta_2$$

for all $(n+m)$-deminsional vectors $\beta = (\beta_0, \beta_1, \beta_2)^T$ and all grid functions ϕ_h. (y_h, z_h)

satisfies

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(6.39)
$$|(y_h, z_h)| < c_{23}(|\beta| + |\phi_h|_h)$$

for a positive constant c_{23} independent of ϵ.

Proof: Let the $n \times n$ matrices T_h be defined by

(6.40)
$$\epsilon D_h(T_h)^- - S_h^-\{(A_h^{11})^-(T_h)^- + (A_h^{12})^-\} = 0, \quad (T_{00})^- = 0$$
$$\epsilon D_h(T_h)^+ - S_h^+\{(A_h^{11})^+(T_h)^+ + (A_h^{12})^+\} = 0, \quad (T_{N0})^+ = 0 \; .$$

(Here $T_h^-((A_h^{11})^-, (A_h^{12})^-)$ and $T_h^+((A_h^{11})^+, (A_{12}^h)^+)$ denote the first n_- and last n_+ rows of $T_h(A_{11}^h, A_{12}^h)$). Because of Lemma (6.19) T_h satisfies $T_h = -(A^{11})_h^{-1}(\tilde{A}^{12})_h + \hat{T}_h + 0(\epsilon)$ and

(6.41)
$$|\hat{T}_{ij}^-| < \prod_{r=0}^{i}(1 + \frac{\kappa h_r}{\epsilon})^{-1}, \; j = 1(1)q, \quad |\hat{T}_{ij}^+| < \prod_{r=i}^{N-1}(1 + \frac{\kappa h_r}{\epsilon})^{-1} \; ,$$
$$j = 0(1)q-1, \; i = 0(1)N-1 \; .$$

We define $u_h = y_h - T_h z_h$ and obtain

$$\epsilon D_{ij}^\pm u_h = S_{ij}^\pm(A_h^{11} u_h) + \phi_{ij}^\pm + h_i S_{ij}^\pm\{(A_h^{11})^\pm T_h^\pm + (A^{12})_h^\pm)D_h z_h\} - \epsilon T_{ij}^\pm D_{ij} z_h$$

(6.42)
$$D_h z_h = S_h^0[A_h^{21} u_h + \{A_h^{22} - A_h^{21}(A_h^{11})^{-1} A_h^{12}\}z_h)] + S_h^0(A_h^{21} \hat{T}_h z_h) + \phi_h^0$$

$$u_{00}^- = \beta_0, \; u_{N0}^+ = \beta_1, \; B^R(-(A^{11})_h^{-1} A_h^{12} z_h, z_h) = \beta_2 - B^R(T_h, u_h, z_h) \; .$$

So for ϵ and H sufficiently small (6.42) can be solved by using contraction and (6.39) holds.

With the aid of Lemma (6.36) we now construct the general solution of $L_{\epsilon h}(y_h, z_h) = \phi_h$ i.e. a solution which depends on $(n+m)$ free parameters. We start with the homogenous problem:

(6.43) Lemma: Let the assumptions of Lemma (6.36) hold. Then there exist solutions $(y_h^\ell, z_h^\ell), (y_h^r, z_h^r)$ of the problems

(6.44)
$$L_{\epsilon h}(y_h^\ell, z_h^\ell) = 0, \; (y_{00}^\ell)^- = n^-, \; (y_{N0}^\ell)^+ = 0, \; z_{N0}^\ell = 0$$
$$L_{\epsilon h}(y_h^r, z_h^r) = 0, \; (y_{00}^r)^- = 0, \; (y_{N0}^r)^+ = n^+, \; z_{00}^r = 0$$

for any n dimensional parameter vector $n = (n^-, n^+)^T$. For any constant $\kappa < c_{23}$

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(where c_{23} is defined as in Lemma (6.36) there exists a $K(\kappa)$ so that

$$|(y_{ij}^{\ell})^-| < K(r) \ \ln^- IP(0,i,\kappa), \quad j = 1(1)q$$

(6.45)

$$|((y_{ij}^{\ell})^+, z_{ij}^{\ell})| < K(\kappa) \ \ln^- I\min\{\varepsilon, P(0,i-1,\kappa), \quad j = 0(1)q-1$$

$$|(y_{ij}^{\Sigma})^+| < K(\kappa) \ \ln^+ IP(i,N-1,\kappa), \quad j = 0, 1, q-1$$

(6.46)

$$|((y_{ij}^{\Sigma})^-, z_{ij}^{\Sigma})| < K(\kappa) \ \ln^+ I\min\{\varepsilon, P(i+1,N-1,\kappa)\}, \quad j = 1(1)q$$

$$P(k,\ell,\kappa) = \prod_{r=k}^{\ell} (1 + \frac{\kappa h_r}{\varepsilon})^{-1}$$

holds independent of ε.

Proof: We define $u_{ij}^- = (y_{ij}^{\ell})^- P(0,i,\kappa)^{-1}$, $v_{ij} = z_{ij}^{\ell} P(0,i,)^{-1}$, $j = 1(1)q$,

$(u_{ij}^{\ell})^+ = (y_{ij}^{\ell})^+ P(0,i-1,\kappa)^{-1}$, $j = 0(1)q-1$, $i = 0(1)N-1$. Then u_h and v_h satisfy

(6.47) $\qquad \varepsilon D_{ij} u_h^- = S_{ij}^-((A_h^{11})^- u_h^- + (A_h^{12})^- v_h) + \kappa u_{i0}^-$

(6.48) $\qquad \varepsilon D_{ij} u_h^+ = S_{ij}^+((A_h^{11})^+ u_h^+ + (A_h^{12})^+ d_h^1 v_h) + \kappa(1 + \frac{\sigma h_i}{\varepsilon})^{-1} u_{iq}^+$

$\qquad \varepsilon D_{ij} v_h = \kappa v_{i0} + \varepsilon S_{ij}^0 (A_h^{21} D_h u_h + A_h^{22} d_h^2 v_h)$

(6.49)

$\qquad u_{00}^- = \eta^-, \ u_{N0}^+ = 0, \ v_{N0} = 0$

$\qquad d_{ij}^1 = (1 + \frac{\kappa h_i}{\varepsilon})^{-1}, \ j = 1(1)q-1, \ d_{i0}^1 = 1, \ d_{ij}^2 = 1, \ j = 1(1)q$

(6.50)

$\qquad d_{i0}^2 = 1 + \frac{\sigma h_i}{\varepsilon}, \ D_{ij} = \mathrm{diag}((d_{ij}^1),\ldots,(d_{ij}^1)^{-1}, \ (d_{ij}^2)^{-1},\ldots,(d_{ij}^2)^{-1}) \ .$

(To derive (6.48) the equations $\omega_{jk}^+ = \omega_{q,q-k}^- - \omega_{q-j,q-k}^-$ and $\omega_{jk}^- = \omega_{q,q-k}^+ - \omega_{q-j,q-k}^+$ have to be used.) Since we can look at (6.49) as a perturbation of the explicit Euler-scheme the terminal value problem for (6.49) is stable. Thus we obtain (using a perturbation argument for (6.47) and (6.48)) that $|(u_h, v_h)| \leq \mathrm{const} |\eta|$ holds if κ is less c_{23}^{-1} (where c_{23} is defned as in Lemma (6.36)). Because of Lemma (6.28)

$|z_h^\ell|_h = 0(\epsilon)$ holds and therefore $|(y_h^\ell)^+|_h$ is also of order $0(\epsilon)$. Therefore (6.45) holds. (6.46) is proved analogously.

Proof of Theorem (3.10): To complete the construction of the general solution of $L_{\epsilon h}(y_h, z_h) = \phi_h$ we first define a solution $(\tilde{y}_h, \tilde{z}_h)$ of the inhomogeneous problem which depends on the m-dimensional free parameter vector $\tilde{\eta}$:

(6.52)
$$L_{\epsilon h}(\tilde{y}_h, \tilde{z}_h) = \phi_h, \quad (\tilde{y}_{00})^- = 0, \quad (\tilde{y}_{N0})^+ = 0, \quad B^R(\tilde{y}_h, \tilde{z}_h) = \tilde{\eta} \ .$$

Let (y_h^ℓ, z_h^ℓ), (y_h^r, z_h^r), η^-, η^+ be defined as in Lemma (5.45). Then $(\tilde{y}_h + y_h^\ell + y_h^r, \tilde{z}_h + z_h^\ell + z_h^r)$ is the general solution of $L_{\epsilon h}(y_h, z_h) = \phi_h$. To prove the existence of a solution of (3.6) and the stability of the associated operator we show that the equation, which arises when we insert the general solution in the boundary conditions (3.4) is soluble. We obtain

(6.53)
$$B_{11}\binom{\eta^-}{\eta^+} + B_{12}(\tilde{y}_{00}, \tilde{y}_{N0}, \tilde{z}_{00}, \tilde{z}_{N0}) + 0(\epsilon|\eta^-, \eta^+|) = \beta_1$$

$$\tilde{\eta} + 0(\epsilon|(\eta^-, \eta^+)|) = \beta_2 \ .$$

This system is invertible and the norm of the inverse is bounded uniformly in ϵ. So there exists a constant c_{24} such that $|(\eta^-, \eta^+, \tilde{\eta})| \leq c_{24}|\beta|$ holds. Therefore $|(y_h, z_h)| \leq \text{const}(|\beta| + |\phi_h|_h)$ holds independent of ϵ and the grid T_H.

(6.54) Proof of Theorem (4.15): We start with constructing an $0(\epsilon)$ approximation of the solution of (3.1): Let $(\tilde{y}_h, \tilde{z}_h)$, $\hat{y}_h$, and (y_h^0, z_h^0) be defined by

(6.55) $\quad F_h^R(\tilde{y}_h, \tilde{z}_h) = 0, \quad b^R(\tilde{y}_h, \tilde{z}_h) = 0, \quad \epsilon D_h \hat{y}_h^- = S_h^-(f_1^-(t_h, \tilde{z}_h)\hat{y}_h^-)$

$$\epsilon D_h \hat{y}_h^+ = S_h^+(f_1^+(t_h, \tilde{z}_h)\hat{y}_h^+)$$

$$\binom{\hat{y}_{00}^-}{\hat{y}_{N0}^+} = -V^{-1}(\tilde{z}_{00}, \tilde{z}_{N0})(W_1(\tilde{z}_{00}, \tilde{z}_{N0})(b_{12}(\tilde{z}_{00}, \tilde{z}_{N0})\binom{\tilde{y}_{00}^+}{\tilde{y}_{N0}^-} + b_2(z_{00}, z_{N0}, 0, 0, 0))$$

$$y_h^0 = \tilde{y}_h + \hat{y}_h, \quad z_h^0 = \tilde{z}_h \ .$$

Here V and W are defined as in (2.7). Because of (2.4) there exists a solution of (5.55). Let u_h and v_h be defined by $u_h = y_h - y_h^0$, $v_h = z_h - z_h^0$. Then u_h and v_h

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satisfy

$$\varepsilon D_h u_h^- = s_h^- (\frac{\partial f^-}{\partial y}(\omega_h)u_h^- + \frac{\partial f^-}{\partial z}(\omega_h)v_h) + H_1(u_h,v_h) + 0(\varepsilon|\hat{y}_h \ell_h)$$

$$\varepsilon D_h u_h^+ = s_h^+ (\frac{\partial f^+}{\partial y}(\omega_h)u_h^+ + \frac{\partial f^+}{\partial z}(\omega_h)v_h) + H_2(u_h,v_h) + 0(\varepsilon|\hat{y}_h \ell_h)$$

$$\varepsilon D_{ij} v_h = s_{ij}^0 (\frac{\partial g}{\partial y}(\omega_h)u_h + \frac{\partial g}{\partial z}(\omega_h)v_h) + H_3(u_h,v_h) + 0(\varepsilon|\hat{y}_{ij}|)$$

$$\frac{\partial (Wb)}{\partial y_h, z_h}(\omega_h)\binom{u_h}{v_h} + H_4(u_h,v_h) + 0(\varepsilon|(u_h,v_h)\ell_h) = 0 \quad .$$

Here ω_h denotes $(t_h, y_h, z_h, \varepsilon)$. $H_i(u_h,v_h)$ satisfies $|H_i(u_h^1,v_h^1) - H_i(u_h^2,v_h^2)|$

$\leq c_{25} \max_{j=1,2} |(u_h^j,v_h^j)| \cdot |(u_h^1-u_h^2,v_h^1-v_h^2)|$, $H_i(0,0) = 0$, $i = 1(1)4$, for some constant c_{25}

independent of ε. Because of Theorem (3.10) we can now use nonlinear Picard-iteration

starting with $(u_h,v_h) = (0,0)$ to show the existence of a solution of (3.1). We obtain

$|(y_h-y_h^0,z_h-z_h^0)| = 0(\varepsilon)$ since the first step of the iteration is of order $0(\varepsilon)$. Keller

(1975) shows, that if a nonlinear operator has a Lipschitz continuous linearization within

a sphere $K_{\rho_0}((y_h^0,z_h^0))$ then the operator is stable in a sphere $K_{\rho_1}((y_h^0,z_h^0))$ and ρ_1 and

the stability constant depend only on the Lipschitz constant and the norm of the inverse of

the linearization of the operator at (y_h^0,z_h^0).

(6.57) Proof of Theorem (4.18): Let $y_0(t)$, $z_0(t)$, $\mu_0(\frac{t}{\varepsilon})$, $\nu_0(\frac{1-t}{\varepsilon})$ be defined as in (2.1),

(2.2) and let $(\tilde{y}_h,\tilde{z}_h)$, $\hat{y}_h$ be defined as above. Since (5.55) is the discrete counterpart

of the reduced problem and we use a method of order $(q+1)$ we obtain

(6.58) $\qquad\qquad |(\tilde{y}_h,\tilde{z}_h) - (\tilde{y}_0(t_h), \tilde{z}_0(t_h))| = 0(H^{q+1})$

by using (2.7) and standard arguments. Because of Lemma (5.19) we obtain for H

sufficiently small

(6.59) $\quad |\hat{y}_{ij}| \leq const \prod_{r=0}^{i} (1 + \frac{\sigma h_r}{\varepsilon})^{-1}$, $j = 1(1)q$, $|\hat{y}_{ij}| \leq const \prod_{r=i}^{N-1} (1 + \frac{\sigma h_r}{\varepsilon})$, $j=0,q-1$.

Estimating $|(y_{ij}^0,z_{ij}^0) - (y_0(t_{ij}),z_0(t_{ij}))|$ by $|(\tilde{y}_h,\tilde{z}_h)-(\tilde{y}_0(t_h),\tilde{z}_0(t_h))| + |\hat{y}_{ij}| + |\mu_0(t_{ij})| + |\nu_0(t_{ij})|$ gives the result.

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(6.60) **Proof of Lemma (3.22):** Let $(y(t_h,\varepsilon), z(t_h,\varepsilon))$ denote the solution of (1.1) restricted on the grid T_H. Then we have

$$(6.61) \qquad \varepsilon D_{ij} y(t_h,\varepsilon) = \int_0^{u_j} \varepsilon y'(t_{i0}+h_i s,\varepsilon)\,ds .$$

If we interpolate $y'(t_{i0}+h_i s,)$ at $u_1,..,u_q$ by a polynomial of degree $q-1$ (which is integrated exactly by $\bar{s}_{ij}$) and use the formula for the remainder term of polynomial interpolation we get

$$(6.62) \qquad |(\ell_{ij}^y)^-| \le \text{const } \varepsilon h_i^q \, |y^{(q+1)}|_{(t_i,t_{i+1})} .$$

Similarly we get $|(\ell_{ij}^y)^+| = 0(\varepsilon h_i^q |y^{(q+1)}|_{(t_i,t_{i+1})})$, $|\ell_{ij}^z| = 0(h_i^{q+1}|z^{(q+2)}|_{(t_i,t_{i+1})})$. It follows from the asymptotic expansion (2.1), (2.2) that $y^{(q+1)}$, $z^{(q+2)}$ satisfy

$$(6.63) \qquad |(y^{(q+1)}(t,\varepsilon),\ z^{(q+2)}(t,\varepsilon))| \le \text{const}(1 + \frac{\exp(\frac{-\lambda_- t}{\varepsilon}) + \exp(\lambda_+ \frac{t-1}{\varepsilon})}{\varepsilon^{q+1}}) .$$

(6.64) **Proof of Theorem (4.25):** Let $y(t,\varepsilon)$ be defined by

$$(6.65) \qquad \varepsilon \hat{y}' = A(t,z_0(t,\varepsilon))\hat{y},\ \hat{y}^-(0,\varepsilon) = \mu_0^-(0),\ \hat{y}^+(1,\varepsilon) = \nu_0^+(0)$$

where z_0, μ_0, ν_0 are defined as in (2.1). As it is easy to show $\hat{y}$, μ and ν satisfy $|\hat{y}^- - \mu_0^-|_{(0,1)} = 0(\varepsilon)$, $|\hat{y}^+ - \nu_0^+|_{(0,1)} = 0(\varepsilon)$. Let now y_h and z_h be defined as in the proof of Theorem (4.15). Since $|z_h - z_0(t_h,\varepsilon)|_h = 0(H^{q+1})$ holds and the initial and terminal value problems in (5.55) for $\hat{y}_h$ are stable we obtain

$$|\hat{y}_h - \hat{y}(t_h,\varepsilon)|_h \le \text{const}(|\tilde{z}_h - \tilde{z}_0(t_h,\varepsilon)|_h +$$

(6.66)

$$+ \max(\varepsilon h_i^q |\hat{y}^{(q+1)}|_{(t_i,t_{i+1})}) : i = 0(1)N-1 = 0(\delta)$$

if we choose the mesh according to (3.24). Summarizing (6.65) and (6.66) we obtain that $|(y_h^0, z_h^0) - (y_0(t_h,\varepsilon), z_0(t_h,\varepsilon))|_h = 0(\varepsilon+\delta)$ holds, where (y_0,z_0) is defined as in Theorem (3.18). Therefore $|(y_h,z_h) - (y(t_h,\varepsilon), z(t_h,\varepsilon))| = 0(\varepsilon+\delta)$ holds. So for ε and δ sufficiently small $(y(t_h,\varepsilon)z(t_h,\varepsilon))$ lies within the ball $K_\rho(y_h,z_h)$. So we can apply Theorem (3.15) and obtain that $|(y_h,z_h) - (y(t_h), z(t_h))| \le \text{const } \delta$ holds independent of ε.

Proof of Theorem (4.27): We choose $h_i = \epsilon \delta^{\frac{1}{q}} \exp\left(\frac{\lambda t_i}{q\epsilon}\right)$ as long as $h_i < \delta^{\frac{1}{q}}$ holds i.e. as long as $t_i < \frac{\epsilon q}{\lambda} \ln(\frac{1}{\epsilon})$ holds. By setting $\bar{h}_i = \frac{\lambda h_i}{q\epsilon}$, $\bar{t}_i = \frac{t_i \lambda}{q\epsilon}$ Step 1 in (4.25) becomes

Choose $\bar{h}_i = \alpha e^{\bar{t}_i}$ as long as $\bar{t}_i < \ln(\frac{1}{\epsilon})$ holds. (With $\alpha = \frac{\lambda \delta^{\frac{1}{q}}}{q}$.) Given ϵ_0 and α assume that we would need N points, i.e. $t_N > \ln\frac{1}{\epsilon_0}$ holds. If we add 1 point $t_{N+1} > +\ln(\frac{1}{\epsilon_0} \exp(\frac{\alpha}{\epsilon_0}))$. Therefore $N+1$ is the number of points we would need to solve the same problem with $\epsilon_1 = \epsilon_0 e^{-\frac{\alpha}{\epsilon_0}}$. Computing the sequence $\epsilon_{n+1} = \epsilon_n \exp(-\frac{\alpha}{\epsilon_n})$ shows that for $\alpha > 10^{-3}$ it decreases from $\epsilon = 10^{-2}$ to $\epsilon = 10^{-18}$ in 3 steps.

To obtain estimate (4.29) we state

$$h_i > \epsilon \delta^{\frac{1}{q}} \exp(i\alpha)$$

$$t_i > \epsilon \delta^{\frac{1}{q}} \sum_{j=0}^{i-1} \exp(j\alpha) = \epsilon \delta^{\frac{1}{q}} \frac{e^{i\alpha}-1}{e^{\alpha}-1} \ .$$

Therefore

$$N < \frac{1}{\alpha} \ln\left(\ln(\frac{1}{\epsilon}) \frac{e^{\alpha}-1}{\alpha} + 1 \right)$$

holds.

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A numerical method for boundary value problems for quasilinear systems of singularly perturbed ordinary differential equations is presented. The method is based on collocation with polynomial splines. The stability properties of the associated difference operator are examined and a stepsize algorithm to achieve a certain over-all accuracy is developed. The number of gridpoints required by the algorithm is estimated.

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